GENERALIZATION OF A RESULT OF MATSUO AND CHEREDNIK TO THE CALOGERO-SUTHERLAND-MOSER INTEGRABLE MODELS WITH EXCHANGE TERMS*

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Abstract

A few years ago, Matsuo and Cherednik proved that from some solutions of the Knizhnik-Zamolodchikov (KZ) equations, which first appeared in conformal field theory, one can obtain wave functions for the Calogero integrable system. In the present communication, it is shown that from some solutions of generalized KZ equations, one can construct wave functions, characterized by any given permutational symmetry, for some Calogero-Sutherland-Moser integrable models with exchange terms. Such models include the spin generalizations of the original Calogero and Sutherland ones, as well as that with δ-function interaction.

1 Introduction

The Calogero integrable system [1] consists of N nonrelativistic particles on the line interacting through a two-body potential of the inverse square type,

$$V_C^{(k)} = k(k-1) \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{(x_i - x_j)^2}. \tag{1}$$

Together with its trigonometric, hyperbolic, and elliptic generalizations [2]–[5], it constitutes the Calogero-Sutherland-Moser (CSM) integrable model, which is related to the root systems of $\mathcal{A}_{N-1}$ algebras [6] and is founding some interesting applications in field-theoretical and condensed-matter contexts.

A breakthrough in the study of CSM systems occurred some three years ago when Polychronakos [7] and Brink *et al* [8] independently introduced an exchange operator formalism, leading to a set of N commuting first-order differential operators, known in the mathematical literature as Dunkl operators [9]. Such operators led to Hamiltonians with exchange terms, connected with the spin generalizations

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of the CSM systems [10]. Spectra and wave functions of the latter can be obtained by simultaneously diagonalizing the N commuting Dunkl operators.

At approximately the same time, another approach to CSM systems made its appearance. It was based upon the Knizhnik-Zamolodchikov (KZ) equations, previously introduced in conformal field theory [11]. The latter can be written as

$$\partial_i \Phi = \left(k \sum_{j \neq i} \frac{P^{(ij)}}{x_i - x_j} + \lambda^{(i)} \right) \Phi, \qquad i = 1, 2, \ldots, N, \tag{2}$$

where $\Phi = \Phi(x_1, x_2, \ldots, x_N)$ takes values in the tensor product $V \otimes V \otimes \cdots \otimes V = V^{\otimes N}$ of some N-dimensional vector space V, $P^{(ij)}$ is the permutation of the i^{th} and j^{th} factors, $\lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$ is a diagonal matrix considered as a parameter, $\lambda^{(i)}$ is the operator in $V^{\otimes N}$ acting as λ on the i^{th} factor and identically on all other factors.

Matsuo [12] and Cherednik [13] indeed showed that if one considers solutions of (2) that can be written as

$$\Phi = \sum_{\sigma \in S_N} \Phi_\sigma e_\sigma, \qquad e_\sigma = e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(N)}, \tag{3}$$

where S_N is the symmetric group, and e_k denotes a column vector with entry 1 in row k and zeroes everywhere else, then the symmetric and antisymmetric combinations

$$\varphi^{[N]} = \sum_{\sigma \in S_N} \Phi_\sigma, \qquad \varphi^{[1^N]} = \sum_{\sigma \in S_N} (-1)^\sigma \Phi_\sigma, \tag{4}$$

of the components Φ_σ of function (3), where $(-1)^\sigma$ denotes the parity of permutation σ, are eigenfunctions with eigenvalue $-\sum_i \lambda_i^2$ of the Calogero Hamiltonian,

$$\left(-\Delta + V_C^{(k)} + \sum_i \lambda_i^2 \right) \varphi^{[N]} = \left(-\Delta + V_C^{(k+1)} + \sum_i \lambda_i^2 \right) \varphi^{[1^N]} = 0, \tag{5}$$

corresponding to parameters k and $k+1$, respectively.

In the present communication, we report on some new results [14] extending those of Matsuo and Cherednik to some CSM systems with exchange terms, thereby providing new links between the latter and some generalized KZ equations. Such equations are reviewed in Sec. 2, and used in Sec. 3 to construct wave functions for CSM systems with exchange terms. Section 4 contains the conclusion.

2 Generalized Knizhnik-Zamolodchikov equations

Let us generalize system (2) into

$$\partial_i \Phi = \left(\sum_{j \neq i} \left(f_{ij}(x_i - x_j) P^{(ij)} + c\, T^{(ij)} \right) + \lambda^{(i)} \right) \Phi, \qquad i = 1, 2, \ldots, N, \tag{6}$$

where all symbols keep the same meaning as in (2), but $k(x_i-x_j)^{-1}$ is replaced by a yet undetermined function $f_{ij}(x_i-x_j)$, and there is an additional term proportional to an operator $T^{(ij)}$ acting only on the i^{th} and j^{th} factors, and such that T is the following operator on $V \otimes V$:

$$T = \sum_{k>l} \left(E_{kl} \otimes E_{lk} - E_{lk} \otimes E_{kl}\right). \tag{7}$$

Here E_{kl} denotes the $N \times N$ matrix with entry 1 in row k and column l and zeroes everywhere else (for a previous use of the operator $T^{(ij)}$, see e.g. [15] and references therein).

Let us again restrict ourselves to solutions of (6) that can be written in the form given by (3). For such functions, Eq. (6) is equivalent to the set of equations

$$\partial_i \Phi_\sigma = \sum_{j \neq i} \left(f_{ij}(x_i - x_j) + c\,\tau_\sigma^{(ij)}\right) \Phi_{\sigma \circ p_{ij}} + \lambda_{\sigma(i)} \Phi_\sigma, \qquad i = 1, 2, \ldots, N, \tag{8}$$

where σ is an arbitrary permutation of S_N, p_{ij} denotes the transposition of i and j, and $\tau_\sigma^{(ij)} \equiv \operatorname{sgn}\big(\sigma(i) - \sigma(j)\big)$ satisfies the relations

$$\begin{aligned}
&\tau_\sigma^{(ij)} = -\tau_{\sigma\circ p_{ij}}^{(ij)} = -\tau_\sigma^{(ji)}, \qquad \tau_{\sigma\circ p_{ij}}^{(ik)} = \tau_\sigma^{(jk)}, \qquad \tau_{\sigma\circ p_{ij}}^{(kl)} = \tau_\sigma^{(kl)}, \\
&\tau_\sigma^{(ij)}\tau_\sigma^{(ik)} + \tau_\sigma^{(jk)}\tau_\sigma^{(ji)} + \tau_\sigma^{(ki)}\tau_\sigma^{(kj)} = 1,
\end{aligned} \tag{9}$$

for any $i \neq j \neq k \neq l$. In deriving Eq. (8), use has been made of the properties that the operators $P^{(ij)}$, $T^{(ij)}$, and $\lambda^{(i)}$ transform the components Φ_σ into $\Phi_{\sigma\circ p_{ij}}$, $\tau_\sigma^{(ij)}\Phi_{\sigma\circ p_{ij}}$, and $\lambda_{\sigma(i)}\Phi_\sigma$, respectively.

It can be easily shown [14] that the integrability conditions of (8), i.e., $\partial_j\partial_i\Phi_\sigma = \partial_i\partial_j\Phi_\sigma$ for any i, $j = 1, 2, \ldots, N$, and any $\sigma \in S_N$, are satisfied if and only if

$$f_{ij}(u) = -f_{ji}(-u), \tag{10}$$

for any i, j, such that $i < j$, and

$$f_{ij}(u) f_{jk}(v) - f_{ik}(u+v)\left[f_{ij}(u) + f_{jk}(v)\right] = -c^2, \tag{11}$$

for any i, j, k, such that $1 \leq i < j < k \leq N$.

Equation (11) looks like a functional equation first considered by Sutherland [4], and solved by Calogero [3] through a small-x expansion. A similar procedure can be used here to derive all the solutions of (11) that are odd and meromorphic in a neighbourhood of the origin [14]. Among the latter, one finds

$$f_{ij}(u) = f_{ji}(u) = F(u), \qquad 1 \leq i < j \leq N, \tag{12}$$

where $F(u)$ denotes the function

$$F(u) = \begin{cases} k\omega \coth \omega u & \text{if } c^2 = k^2\omega^2 > 0, \\ k/u & \text{if } c^2 = 0, \\ k\omega \cot \omega u & \text{if } c^2 = -k^2\omega^2 < 0, \end{cases} \tag{13}$$

and N may take any value such that $N \geq 3$. For $c^2 = 0$, function (12) is the unique solution of Eq. (11) that is odd and meromorphic in a neighbourhood of the origin, while for $c^2 \neq 0$, there is another solution, for which not all $f_{ij}(u)$'s are equal, and which will not play any role in the next Section.

Eq. (11) also has some singular solutions, such as

$$f_{ij}(u) = f_{ji}(u) = c\,\mathrm{sgn}(u) = c\,[\theta(u) - \theta(-u)], \qquad 1 \leq i < j \leq N, \tag{14}$$

where $\theta(u)$ denotes the Heaviside function.

3 Generalization of the result of Matsuo and Cherednik

From a set of $N!$ functions $\Phi_\sigma(x_1, \ldots, x_N)$, $\sigma \in S_N$, satisfying Eq. (8), one can construct in general $N!$ functions $\varphi^{[f]}_{rs}(x_1, \ldots, x_N)$, defined by

$$\varphi^{[f]}_{rs} = \sum_{\sigma \in S_N} V^{[f]}_{rs}(\sigma)\Phi_\sigma, \tag{15}$$

where $[f] \equiv [f_1 f_2 \ldots f_N]$ runs over all N-box Young diagrams, r and s label the standard tableaux associated with $[f]$, arranged in lexicographical order, and $V^{[f]}_{rs}(\sigma)$ denotes Young's orthogonal matrix representation of S_N [16]. For $[f] = [N]$ or $\left[1^N\right]$, since $V^{[N]}(\sigma) = 1$ and $V^{[1^N]}(\sigma) = (-1)^\sigma$, functions (15) reduce to those considered by Matsuo and Cherednik and given in (4).

By using the previous equations, as well as some elementary results about transpositions,

$$p_{ik} \circ p_{ij} = p_{ij} \circ p_{jk} = p_{jk} \circ p_{ik}, \tag{16}$$

and matrix representations,

$$V^{[f]}_{rs}(\sigma \circ \sigma') = \sum_t V^{[f]}_{rt}(\sigma) V^{[f]}_{ts}(\sigma'), \qquad V^{[f]}_{rs}(1) = \delta_{r,s}, \tag{17}$$

it is straightforward to show [14] that the functions $\varphi^{[f]}_{rs}$ satisfy the system of equations

$$\begin{aligned} \partial_i \varphi^{[f]}_{rs} &= \sum_{j \neq i} f_{ij} \sum_t \varphi^{[f]}_{rt} V^{[f]}_{ts}(p_{ij}) - c \sum_{j \neq i} \sum_t \left(\sum_\sigma \tau^{(ij)}_\sigma V^{[f]}_{rt}(\sigma)\Phi_\sigma \right) V^{[f]}_{ts}(p_{ij}) \\ &\quad + \sum_\sigma \lambda_{\sigma(i)} V^{[f]}_{rs}(\sigma)\Phi_\sigma, \qquad i = 1, 2, \ldots, N, \end{aligned} \tag{18}$$

and that their Laplacian is given by

$$\Delta \varphi^{[f]}_{rs} = \left(\sum_{\substack{i,j \\ i \neq j}} \left(f^2_{ij}(x_i - x_j) + (\partial_i f_{ij}(x_i - x_j))\, K_{ij} - c^2 \right) + \sum_i \lambda^2_i \right) \varphi^{[f]}_{rs}. \tag{19}$$

In (19), $K_{ij} = K_{ji}$, $1 \leq i < j \leq N$, are some operators, whose action on $\varphi^{[f]}_{rs}$ is defined by

$$K_{ij}\varphi^{[f]}_{rs} = \sum_t \varphi^{[f]}_{rt} V^{[f]}_{ts}(p_{ij}). \tag{20}$$

In the special cases where $[f] = [N]$ or $\left[1^N\right]$, the operators K_{ij} behave as I or $-I$, respectively. Hence, for f_{ij} given by (12) and (13), where $c^2 = 0$, Eq. (19) reduces to Eq. (5), i.e., Matsuo and Cherednik's result. Let us emphasize that Eq. (19) is also valid in the mixed-symmetry cases, and for any function $\varphi^{[f]}_{rs}$ constructed from any solution of (8) via transformation (15).

As for $[f] \neq [N]$, $\left[1^N\right]$, the operators K_{ij} have a rather complicated effect on the functions $\varphi^{[f]}_{rs}$, it is convenient to restrict the latter so that for any $i < j$, K_{ij} may be interpreted as a permutation operator acting on the variables x_i and x_j, and leaving the remaining variables x_k unchanged,

$$K_{ij}x_j = x_i K_{ij}, \qquad K_{ij}x_k = x_k K_{ij} \qquad k \neq i, j. \tag{21}$$

From (20), it results that the conditions to be fulfilled by $\varphi^{[f]}_{rs}$ are

$$\begin{aligned}&\varphi^{[f]}_{rs}(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N)\\ &\quad = \sum_t \varphi^{[f]}_{rt}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) V^{[f]}_{ts}(p_{ij}), \qquad 1 \leq i < j \leq N.\end{aligned} \tag{22}$$

In terms of the components Φ_σ of (3), such conditions amount to

$$\begin{aligned}&\Phi_\sigma(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N)\\ &\quad = \Phi_{\sigma \circ p_{ij}}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N), \qquad 1 \leq i < j \leq N,\end{aligned} \tag{23}$$

for any $\sigma \in S_N$. By differentiating both sides of (23) with respect to x_k and using (8) to calculate the derivatives, one finds that Eqs. (8) and (23) are compatible if and only if all functions $f_{ij}(u)$, $i \neq j$, coincide, hence in cases such as (12) and (14).

For instance, when f_{ij} is given by (12) and (13), where $c^2 > 0$ (hyperbolic case), Eq. (19) becomes

$$\left(-\Delta + \omega^2 \sum_{\substack{i,j\\ i\neq j}} \left(\operatorname{csch} \omega(x_i - x_j)\right)^2 k(k - K_{ij}) + \sum_i \lambda_i^2 \right) \varphi^{[f]}_{rs} = 0, \tag{24}$$

with K_{ij} defined by (21). Such an equation (as well as similar results for the remaining cases) shows that from any solution of type (3), (23) of the KZ equations (6), with f_{ij} given in (12), one can obtain eigenfunctions $\varphi^{[f]}_{rs}$ of the CSM Hamiltonians [1]–[5] with exchange terms [10], which are characterized by any given permutational symmetry $[f]$ under particle coordinate exchange. To obtain wave functions describing an N-boson (resp. N-fermion) system, it only remains to combine $\varphi^{[f]}_{rs}$

with a spin function transforming under the same (resp. conjugate) irreducible representation $[f]$ (resp. $[\tilde{f}]$) under exchange of the spin variables. A similar result is valid for the Hamiltonian with delta-function interactions [17], corresponding to the functions f_{ij} given in (14).

4 Conclusion

By extending the Matsuo and Cherednik results to some integrable models with exchange terms and wave functions of any permutational symmetry, we showed that there exists a strong interplay between such models and (generalized) KZ equations. As already noted by Brink and Vasiliev [18], this also hints at some deep relationship between the latter and Dunkl operators.

Whether some results similar to those presented here also hold true for elliptic CSM models, and for integrable models related to root systems of algebras different from $\mathcal{A}_{N-1}$ [6], remains an interesting open question.

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